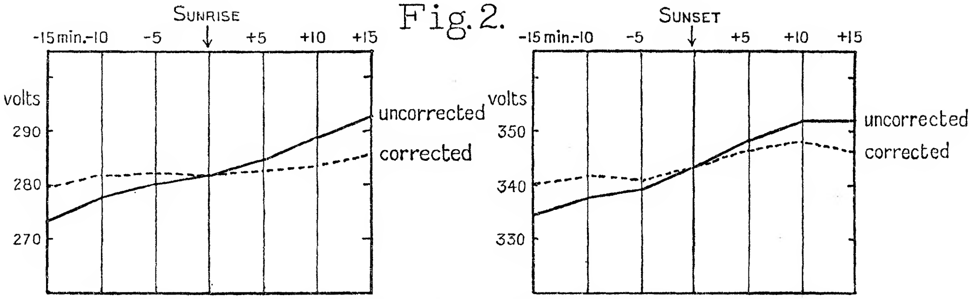
Fig. 2.



Mean Potential Variation at Sunrise and Sunset (Kew, 1912 and 1914).

It is hoped that a detailed study of the synchronous variation of the atmospheric electrical quantities with the integral intensity of "strays" in wireless phenomena may be possible at Kew Observatory in the near future.

On the Motion of Solids in a Liquid Possessing Vorticity.

By J. PROUDMAN.

(Communicated by Prof. H. Lamb, F.R.S. Received March 28, 1916.)

1. The following investigations were undertaken at the suggestion of Mr. G. I. Taylor as an extension of some of his own.*

The equations of motion of any system of solids in a liquid which moves without vorticity were obtained by Lord Kelvin by a method based on that of ignorable co-ordinates in ordinary dynamics. They are of Lagrange's type, and were afterwards obtained by various writers† by an adaptation to hydrodynamics of Lagrange's original method.

Lamb's work leads to some hydrokinematical relations which we shall put in evidence below.

For the rotational problems discussed in the present paper, an attempt to obtain a Lagrangian function has only been successful in one case. This case is that to which Taylor's principal results refer, viz., the two-dimensional motion, with uniform vorticity, which at infinity reduces to a rotation about an axis. That a Lagrangian function exists in this case is obvious from Taylor's results.

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* An account of these investigations has not yet been published. The subsequent references are to a MS. which Mr. Taylor very kindly showed the author.

† See Lamb, 'Hydrodynamics,' 3rd Edit., Art. 136.

In other cases, instead of bringing into the equations the effects of the inertia of the solids, restriction is made to the finding of the generalised components of liquid pressure on the solids. These are denoted by Q_r, where

$$Q_r = \Sigma \iint p \frac{\partial \phi_r}{\partial n} dS, \tag{1}$$

in the notation of the next section, p denoting the pressure intensity.

For two-dimensional problems with uniform vorticity, this is immediately effected by an extension of the ordinary pressure formula.

For three-dimensional problems, constant uniform vorticity becomes impossible, and the motion of the liquid depends not only on the instantaneous motion of the solids, but also on its past state. Throughout the consideration of these problems, the motion is referred to axes which rotate with constant angular velocity. The determination of the pressure is first reduced to the solution of a Neumann's problem, and the method is then applied to calculate the pressure components in any case of initial motion. The results are in agreement with the conclusions and experiments of Taylor.

Two general problems of continued motion are then formally solved by means of infinite power series in the time. The first of these problems concerns the evolution of any given small relative motion inside a rotating envelope; the second concerns the motion of a body without relative rotation in a liquid which extends to infinity and is there at relative rest. The coefficients in the power series are derived from a certain sequence of functions of position, and each function may be found from the two immediately preceding ones by the solution of a Neumann's problem.

The convergence of the power series is not here established, but for the special case of a spherical boundary the series have been shown to be highly convergent. The demonstration of this is reserved for a future paper, in which it is hoped to give a solution of the small motion of a sphere in a form which is numerically intelligible. The functions of position, above mentioned, are at present being evaluated in this case.

To render the problems mathematically determinate in as simple a way as possible, motion at the outer boundary or infinity is prescribed. It is supposed that, if necessary, this prescribed state is maintained by external agency not here considered.

Hydrokinematical Relations.

2. Let x, y, z be Cartesian co-ordinates referred to fixed axes, and let

$$q_1, q_2, \dots, q_n,$$

be independent co-ordinates of the Lagrangian type completely specifying at time t the positions of the solids. Also, let

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_n,$$

be functions of x, y, z, t and $q_1, q_2, \ldots, q_n$, such that ϕ_r is the velocity potential of the irrotational motion in the case for which

$$q^{\cdot}_r = 1, \qquad q^{\cdot}_s = 0 \qquad (s \neq r),$$

there being no motion at infinity, if the liquid be not enclosed in an envelope.

When the motion is irrotational, the velocity potential ϕ is given by

$$\phi = q^{\cdot}_1\phi_1 + q^{\cdot}_2\phi_2 + \ldots + q^{\cdot}_n\phi_n, \tag{2}$$

but we shall also make use of the functions $\phi_1, \ldots, \phi_n$, when the motion is not irrotational.

Let $\phi^{\cdot}$ denote the rate of increase of ϕ for fixed values of x, y, z, and q the speed of a particle of liquid. Then it easily follows from the work of Lamb (*loc. cit.*) that

$$\Sigma \iint (\phi^{\cdot} - \tfrac{1}{2}q^2)\frac{\partial\phi_r}{\partial n}\,dS = \frac{d}{dt}\Sigma\iint \phi\frac{\partial\phi_r}{\partial n}\,dS - \tfrac{1}{2}\frac{\partial}{\partial q_r}\Sigma\iint\phi\frac{\partial\phi}{\partial n}\,dS, \tag{3}$$

where $\partial/\partial n$ denotes differentiation along the outward-drawn normal to the surface of a solid, where the surface integrals are taken over the surfaces of the solids, and the summation extends to all the solids.

To simplify matters it is here supposed that the liquid extends to infinity and is there at rest.

Now on transforming the surface integrals on the right-hand side of (3) into volume integrals taken throughout the liquid, and then differentiating and re-transforming to surface integrals, we see that

$$\frac{d}{dt}\Sigma\iint\phi\frac{\partial\phi_r}{\partial n}\,dS = \Sigma\iint\left\{\phi^{\cdot}\frac{\partial\phi_r}{\partial n} + \phi^{\cdot}_r\frac{\partial\phi}{\partial n} - \left(\frac{\partial\phi}{\partial x}\frac{\partial\phi_r}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\phi_r}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\phi_r}{\partial z}\right)\frac{\partial\phi}{\partial n}\right\}dS, \tag{4}$$

$$\tfrac{1}{2}\frac{\partial}{\partial q_r}\Sigma\iint\phi\frac{\partial\phi}{\partial n}\,dS = \Sigma\iint\left(\frac{\partial\phi}{\partial q_r}\frac{\partial\phi}{\partial n} - \tfrac{1}{2}q^2\frac{\partial\phi_r}{\partial n}\right)dS, \tag{5}$$

so that (3) gives

$$\Sigma\iint\left\{\phi^{\cdot} - \left(\frac{\partial\phi}{\partial x}\frac{\partial\phi_r}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\phi_r}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\phi_r}{\partial z}\right) - \frac{\partial\phi}{\partial q_r}\right\}\frac{\partial\phi}{\partial n}\,dS + \Sigma\iint q^2\frac{\partial\phi_r}{\partial n}\,dS = 0. \tag{6}$$

Again,

$$\frac{\partial\phi}{\partial n} = \frac{\partial\phi_1}{\partial n}q^{\cdot}_1 + \ldots + \frac{\partial\phi_n}{\partial n}q^{\cdot}_n,$$

$$\frac{\partial\phi}{\partial q_r} = \frac{\partial\phi_1}{\partial q_r}q^{\cdot}_1 + \ldots + \frac{\partial\phi_n}{\partial q_r}q^{\cdot}_n,$$

$$\phi^{\cdot}_r = \frac{\partial\phi_r}{\partial q_1}q^{\cdot}_1 + \ldots + \frac{\partial\phi_r}{\partial q_n}q^{\cdot}_n,$$

$$\frac{\partial\phi}{\partial x}\frac{\partial\phi_r}{\partial x}+\frac{\partial\phi}{\partial y}\frac{\partial\phi_r}{\partial y}+\frac{\partial\phi}{\partial z}\frac{\partial\phi_r}{\partial z}=\left(\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_r}{\partial x}+\frac{\partial\phi_1}{\partial y}\frac{\partial\phi_r}{\partial y}+\frac{\partial\phi_1}{\partial z}\frac{\partial\phi_r}{\partial z}\right)\dot{q}_1+\dots$$
$$+\left(\frac{\partial\phi_n}{\partial x}\frac{\partial\phi_r}{\partial x}+\frac{\partial\phi_n}{\partial y}\frac{\partial\phi_r}{\partial y}+\frac{\partial\phi_n}{\partial z}\frac{\partial\phi_r}{\partial z}\right)\dot{q}_n,$$

$$q^2=\left(\frac{\partial\phi_1}{\partial x}\dot{q}_1+\dots+\frac{\partial\phi_n}{\partial x}\dot{q}_n\right)^2+\left(\frac{\partial\phi_1}{\partial y}\dot{q}_1+\dots+\frac{\partial\phi_n}{\partial y}\dot{q}_n\right)^2+\left(\frac{\partial\phi_1}{\partial z}\dot{q}_1+\dots+\frac{\partial\phi_n}{\partial z}\dot{q}_n\right)^2,$$

so that, on equating to zero the coefficient of $\dot{q}_s\dot{q}_t$ in (6), we obtain

$$\Sigma\iint\left[\left\{\left(\frac{\partial\phi_r}{\partial q_s}-\frac{\partial\phi_s}{\partial q_r}\right)-\left(\frac{\partial\phi_r}{\partial x}\frac{\partial\phi_s}{\partial x}+\frac{\partial\phi_r}{\partial y}\frac{\partial\phi_s}{\partial y}+\frac{\partial\phi_r}{\partial z}\frac{\partial\phi_s}{\partial z}\right)\right\}\frac{\partial\phi_t}{\partial n}\right.$$
$$+\left\{\left(\frac{\partial\phi_r}{\partial q_t}-\frac{\partial\phi_t}{\partial q_r}\right)-\left(\frac{\partial\phi_r}{\partial x}\frac{\partial\phi_t}{\partial x}+\frac{\partial\phi_r}{\partial y}\frac{\partial\phi_t}{\partial y}+\frac{\partial\phi_r}{\partial z}\frac{\partial\phi_t}{\partial z}\right)\right\}\frac{\partial\phi_s}{\partial n}$$
$$\left.+2\left\{\frac{\partial\phi_s}{\partial x}\frac{\partial\phi_t}{\partial x}+\frac{\partial\phi_s}{\partial y}\frac{\partial\phi_t}{\partial y}+\frac{\partial\phi_s}{\partial z}\frac{\partial\phi_t}{\partial z}\right\}\frac{\partial\phi_r}{\partial n}\right]dS=0. \tag{7}$$

This relation remains valid if either s or t is equal to r or if they are equal to one another; in both these cases it takes the same simpler form.

As an illustration of the relation (7), let us take the case of a single solid with q_1, q_2, q_3 the Cartesian co-ordinates of a point fixed in it. Then on the surface of the solid we have

$$-\frac{\partial\phi_1}{\partial n}=l,\qquad -\frac{\partial\phi_2}{\partial n}=m,\qquad -\frac{\partial\phi_3}{\partial n}=n,$$

l, m, n denoting the direction cosines of the outward-drawn normal, and

$$\frac{\partial}{\partial q_1}=-\frac{\partial}{\partial x},\qquad \frac{\partial}{\partial q_2}=-\frac{\partial}{\partial y},\qquad \frac{\partial}{\partial q_3}=-\frac{\partial}{\partial z},$$

when applied to ϕ_1, ϕ_2, ϕ_3. Taking $r=1$, $s=t=2$, in (7) we have

$$\iint\left[m\left\{\left(\frac{\partial\phi_2}{\partial x}-\frac{\partial\phi_1}{\partial y}\right)-\left(\frac{\partial\phi_1}{\partial x}\frac{\partial\phi_2}{\partial x}+\frac{\partial\phi_1}{\partial y}\frac{\partial\phi_2}{\partial y}+\frac{\partial\phi_1}{\partial z}\frac{\partial\phi_2}{\partial z}\right)\right\}\right.$$
$$\left.+l\left\{\left(\frac{\partial\phi_2}{\partial x}\right)^2+\left(\frac{\partial\phi_2}{\partial y}\right)^2+\left(\frac{\partial\phi_2}{\partial z}\right)^2\right\}\right]dS=0, \tag{8}$$

the truth of which is easily verified by Green's theorem. It is interesting to notice the connection between this relation and those used by Prof. Lamb* to derive from the pressure formula the equations of motion for this particular case.

Two-Dimensional Problems with Uniform Vorticity.

3. Let ψ denote the stream function and ω the vorticity, so that

$$\nabla_1^2\psi=2\omega, \tag{9}$$

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* "On the Forces Experienced by a Solid Moving through a Liquid," 'Quart. Journ. Math.,' vol. 19, p. 66 (1883).

where ∇_1^2 denotes $\partial^2/\partial x^2+\partial^2/\partial y^2$, x, y being Cartesian co-ordinates in the plane of the motion.

Since
$$\nabla_1^2\dot{\psi} = 0, \tag{10}$$
we can find a function $\dot{\phi}$, such that $\dot{\phi}$, $\dot{\psi}$ are conjugate functions, *i.e.* $\dot{\phi}+i\dot{\psi}$ is a function of $x+iy$.

Now the equations of motion may be written in the form
$$\left.\begin{aligned}\frac{\partial u}{\partial t}-2\omega v &= -\frac{\partial \mathrm{P}}{\partial x},\\ \frac{\partial v}{\partial t}+2\omega u &= -\frac{\partial \mathrm{P}}{\partial y},\end{aligned}\right\} \tag{11}$$
where
$$\mathrm{P} = p/\rho+\tfrac{1}{2}q^2+\Omega, \tag{12}$$
Ω being the potential of the external forces.

Since
$$u = -\frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\psi}{\partial x},$$
we have
$$\frac{\partial u}{\partial t} = -\frac{\partial\dot{\psi}}{\partial y} = -\frac{\partial\dot{\phi}}{\partial x},$$
$$\frac{\partial v}{\partial t} = \frac{\partial\dot{\psi}}{\partial x} = -\frac{\partial\dot{\phi}}{\partial y},$$
and the equations of motion may be written
$$\left.\begin{aligned}-\frac{\partial\dot{\phi}}{\partial x}-2\omega\frac{\partial\psi}{\partial x} &= -\frac{\partial \mathrm{P}}{\partial x},\\ -\frac{\partial\dot{\phi}}{\partial y}-2\omega\frac{\partial\psi}{\partial y} &= -\frac{\partial \mathrm{P}}{\partial y},\end{aligned}\right\} \tag{13}$$
From these we immediately obtain the integral
$$p/\rho = \dot{\phi}+2\omega\psi-\tfrac{1}{2}q^2-\Omega, \tag{14}$$
it being supposed that the arbitrary additive function of t in $\dot{\phi}$ is chosen so that (14) is valid as it stands.

From (14) the particular results of Taylor, for a circular cylinder in a liquid whose motion at infinity is one of shearing, may be very easily reproduced.

Now let us refer the motion to axes which rotate with constant angular velocity equal to ω. Let u', v' refer to the motion relative to these axes, and let ψ' be the relative stream function, so that
$$u' = -\frac{\partial\psi'}{\partial y}, \quad v' = \frac{\partial\psi'}{\partial x}, \quad \nabla_1^2\psi' = 0. \tag{15}$$

The equations (11) remain valid with accented variables, if we take, instead of (12),
$$\mathrm{P}' = p/\rho+\tfrac{1}{2}q'^2+\Omega-\tfrac{1}{2}\omega^2r^2, \tag{16}$$

q' denoting the relative speed of a liquid particle and r the distance from the axis of rotation. Taking ϕ' so that ϕ', ψ' are conjugate functions, we obtain, as before, the integral

$$p/\rho = \phi^{\cdot\prime} + 2\omega\psi' - \tfrac{1}{2}q'^2 - \Omega + \tfrac{1}{2}\omega^2 r^2. \tag{17}$$

In this case $\phi' = \phi_1 q^{\cdot}{}_1 + \ldots + \phi_n q^{\cdot}{}_n$.

The foregoing results of the present section were obtained from the lectures of Mr. Herman, who quotes a paper by Prof. Love* as virtually containing them.

4. Taking the case of rotating axes with relative rest at infinity, let us obtain the pressure components on unit length of a system of solid cylinders. We have, from (1) and (17),

$$Q_r = Q_r' + Q_r'' + Q_r''',$$

where

$$\frac{Q_r'}{\rho} = \Sigma \int \left(\phi^{\cdot\prime} - \tfrac{1}{2}q'^2 - \Omega\right) \frac{\partial\phi_r}{\partial n} ds, \tag{18}$$

$$\frac{Q_r''}{\rho} = 2\omega\, \Sigma \int \psi' \frac{\partial\phi_r}{\partial n} ds, \tag{19}$$

$$\frac{Q_r'''}{\rho} = \tfrac{1}{2}\omega^2\, \Sigma \int r^2 \frac{\partial\phi_r}{\partial n} ds. \tag{20}$$

5. We will now obtain the equations of motion for the problem just considered when $\Omega = 0$, by the methods of Prof. Lamb.

We consider only the dynamics of unit length of the system parallel to the axis of rotation.

Let ξ, η be the co-ordinates of a filament of mass m, whether of liquid or of one of the solids, and let X, Y be the components of force acting on it.

We have

$$\Sigma\, m(\xi^{\cdot\cdot}\Delta\xi + \eta^{\cdot\cdot}\Delta\eta) + 2\omega\, \Sigma\, m(\xi^{\cdot}\Delta\eta - \eta^{\cdot}\Delta\xi) - \omega^2 \Sigma\, m(\xi\Delta\xi + \eta\Delta\eta) = \Sigma(X\Delta\xi + Y\Delta\eta), \tag{21}$$

where $\Delta\xi$, $\Delta\eta$ are the components of a "variation," and the summation extends to all filaments of the system.

Now, just as in the case considered by Lamb,

$$\Sigma\, m(\xi^{\cdot\cdot}\Delta\xi + \eta^{\cdot\cdot}\Delta\eta) = \left\{\frac{d}{dt}\left(\frac{\partial T'}{\partial q^{\cdot}{}_1}\right) - \frac{\partial T'}{\partial q_1}\right\}\Delta q_1 + \ldots + \left\{\frac{d}{dt}\left(\frac{\partial T'}{\partial q^{\cdot}{}_n}\right) - \frac{\partial T'}{\partial q_n}\right\}\Delta q_n, \tag{22}$$

where T' is the total kinetic energy of the relative motion, and

$$\Sigma(X\Delta\xi + Y\Delta\eta) = Q_1\Delta q_1 + \ldots + Q_n\Delta q_n, \tag{23}$$

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* "On the Stability of Certain Vortex Motions," 'Lond. Math. Soc. Proc.,' vol. 25, p. 18 (1893).

where $Q_1, Q_2, \ldots, Q_n$, here denote the generalised components of the external forces applied to the solids only.

In the present problem we have also

$$\Sigma m(\xi\Delta\xi+\eta\Delta\eta) = \tfrac{1}{2}\left(\frac{\partial \mathrm{I}}{\partial q_1}\Delta q_1+\ldots+\frac{\partial \mathrm{I}}{\partial q_n}\Delta q_n\right), \tag{24}$$

where I is the variable part of the total moment of inertia of the system about the axis. It may be regarded as formed from solids occupying always the positions of the actual solids but of density $\sigma-\rho$, where σ is the actual density of the solids, in general, of course, variable.

Further, the contribution to

$$2\Sigma m(\dot{\xi}\Delta\eta-\dot{\eta}\Delta\xi), \tag{25}$$

from the solids only, is

$$\left\{\frac{d}{dt}\left(\frac{\partial \mathrm{M}_0'}{\partial \dot{q}_1}\right)-\frac{\partial \mathrm{M}_0'}{\partial q_1}\right\}\Delta q_1+\ldots+\left\{\frac{d}{dt}\left(\frac{\partial \mathrm{M}_0'}{\partial \dot{q}_n}\right)-\frac{\partial \mathrm{M}_0'}{\partial q_n}\right\}\Delta q_n, \tag{26}$$

where M_0' is the moment of momentum, about the axis, of the relative motion of the solids. The contribution from the liquid is

$$-2\rho\iint\left(\frac{\partial\psi}{\partial y}\Delta\eta+\frac{\partial\psi}{\partial x}\Delta\xi\right)d\mathrm{S},$$

taken over all the liquid, which is equal to

$$2\rho\,\Sigma\int\psi\Delta n\,ds,$$

in our customary notation, where Δn denotes the component of variation along the outward-drawn normal to the boundary of a solid.

Now we might express the motion of the solids by means of a stream function; and although this would not be a harmonic function, it could be taken equal to ψ at the surfaces of the solids. If, then, the density of the solids were uniform, and equal to ρ_0, their contribution (26) would be

$$-2\rho_0\,\Sigma\int\psi\Delta n\,ds.$$

From this we see that the actual contribution from the liquid is equal and opposite to that of a set of solids of uniform density ρ, occupying always the positions of the actual solids.

The total contribution to (25) is therefore

$$\left\{\frac{d}{dt}\left(\frac{\partial \mathrm{M}'}{\partial \dot{q}_1}\right)-\frac{\partial \mathrm{M}'}{\partial q_1}\right\}\Delta q_1+\ldots+\left\{\frac{d}{dt}\left(\frac{\partial \mathrm{M}'}{\partial \dot{q}_n}\right)-\frac{\partial \mathrm{M}'}{\partial q_n}\right\}\Delta q_n, \tag{27}$$

where M' is the moment of momentum, about the axis, of the relative motion of solids, occupying always the positions of the actual solids, but of density $\sigma-\rho$.

The equations of motion may therefore be written

$$\frac{d}{dt}\left(\frac{\partial \mathrm{R}}{\partial \dot{q}_r}\right)-\frac{\partial \mathrm{R}}{dq_r}= \mathrm{Q}_r \qquad (r = 1, 2, \ldots n),$$

where

$$\mathrm{R} = \mathrm{T}'+\omega \mathrm{M}'+\tfrac{1}{2}\omega^2 \mathrm{I}, \tag{28}$$

the functions T′, M′, I, Q_r being formed as described above.

Using this to obtain the generalised components of liquid pressure on the solids, we have

$$\mathrm{Q}_r' = -\left\{\frac{d}{dt}\left(\frac{\partial \mathrm{T}_1'}{\partial \dot{q}_r}\right)-\frac{\partial \mathrm{T}_1'}{\partial q_r}\right\}, \tag{29}$$

$$\mathrm{Q}_r'' = \omega\left\{\frac{d}{dt}\left(\frac{\partial \mathrm{M}_1'}{\partial \dot{q}_r}\right)-\frac{\partial \mathrm{M}_1'}{\partial q_r}\right\}, \tag{30}$$

$$\mathrm{Q}_r''' = -\tfrac{1}{2}\omega^2 \mathrm{I}_1, \tag{31}$$

where T_1' is the kinetic energy of the relative motion of the liquid, M_1' and I_1 are the moment of momentum of relative motion and moment of inertia respectively, of solids occupying always the positions of the actual solids, but of density ρ.

The equivalence of (20) and (31) is easily seen, while the equivalence of (18), with $\Omega = 0$, and (29) follows from the relations of § 2. To show the equivalence of (19) and (30), let us express the possible motions of the solids by means of stream functions $\psi_1', \psi_2', \ldots, \psi_n'$, so that on their boundaries we have

$$\frac{\partial \psi_r'}{\partial s} = \frac{\partial \phi_r}{\partial n}.$$

Then

$$\frac{d}{dt}\left(\frac{d\mathrm{M}_1'}{\partial \dot{q}_r}\right)-\frac{\partial \mathrm{M}_1'}{\partial q_r} = 2\Sigma m\left(\dot{\xi}\frac{\partial \eta}{\partial q_r}-\dot{\eta}\frac{\partial \xi}{\partial q_r}\right)$$

$$= -2\rho\Sigma\iint\left(\frac{\partial \psi'}{\partial y}\frac{\partial \psi_r'}{\partial x}-\frac{\partial \psi'}{\partial x}\frac{\partial \psi_r'}{\partial y}\right)d\mathrm{S},$$

the summation and integration extending over the solids only, and this is equivalent to

$$2\rho\Sigma\int \psi'\frac{\partial \psi_r'}{\partial s}\,ds$$

$$= 2\rho\Sigma\int \psi'\frac{\partial \phi_r}{\partial n}\,ds.$$

Three-Dimensional Problems.

6. We shall use Cartesian axes $Oxyz$ rotating about Oz with constant angular velocity ω. Let u, v, w denote the components of the velocity at

x, y, z, relative to these rotating axes. Then the hydrodynamical equations may be written in the form

$$\left.\begin{aligned} \frac{\partial u}{\partial t}-2(v\zeta-w\eta)-2\omega v &= -\frac{\partial \mathrm{P}}{\partial x}, \\ \frac{\partial v}{\partial t}-2(w\xi-u\zeta)+2\omega u &= -\frac{\partial \mathrm{P}}{\partial y}, \\ \frac{\partial w}{dt}-2(u\eta-v\xi) \qquad &= -\frac{\partial \mathrm{P}}{\partial z}, \end{aligned}\right\} \tag{32}$$

where
$$2\xi = \frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \qquad 2\eta = \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \qquad 2\zeta = \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \tag{33}$$

and
$$\mathrm{P} = p/\rho+\tfrac{1}{2}q^2+\Omega-\tfrac{1}{2}\omega^2(x^2+y^2), \tag{34}$$

q now denoting the relative speed.

For the generalised components of liquid pressure on the solids, we have

$$\frac{\mathrm{Q}_r}{\rho} = \Sigma\iint\{\mathrm{P}-\tfrac{1}{2}q^2-\Omega+\tfrac{1}{2}\omega^2(x^2+y^2)\}\frac{\partial\phi_r}{\partial n}\,d\mathrm{S},$$

or
$$\mathrm{Q}_r = \rho\,\Sigma\iint(\mathrm{P}-\tfrac{1}{2}q^2-\Omega)\frac{\partial\phi_r}{\partial n}\,d\mathrm{S}-\tfrac{1}{2}\omega^2\frac{\partial \mathrm{I}_1}{\partial q_r}, \tag{35}$$

where I_1 denotes the moment of inertia, about the axis of rotation, of solids occupying always the positions of the actual solids, but of density ρ.

We shall now show that, for a given state of motion, the determination of the function P may be reduced to a Neumann's problem.

On using the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0,$$

we easily obtain from (32) and (33) at points in the liquid

$$\nabla^2\mathrm{P} = 4(\xi^2+\eta^2+\zeta^2)+4\omega\zeta+u\nabla^2u+v\nabla^2v+w\nabla^2w, \tag{36}$$

where $\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$. Also, at a boundary of the liquid, we have, from (32),

$$\begin{aligned} -\frac{\partial \mathrm{P}}{\partial n} &= l\frac{\partial u}{\partial t}+m\frac{\partial v}{\partial t}+n\frac{\partial w}{\partial t} \\ &\quad -2\{l(v\zeta-w\eta)+m(w\xi-u\zeta)+n(u\zeta-v\xi)\}-2\omega(lv-mu). \end{aligned} \tag{37}$$

The whole of the right-hand sides of (36) and (37) is supposed known, except

$$l\frac{\partial u}{\partial t}+m\frac{\partial v}{\partial t}+n\frac{\partial w}{\partial t}. \tag{38}$$

We shall now show that this also is determined by the instantaneous state of liquid velocities and the velocities and accelerations of the boundary.

Let
$$F(x, y, z, t) = 0,$$
be the equation of a bounding surface, which may be the surface of one of the solids. Then, if
$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z},$$
we shall have
$$\frac{D^s}{Dt^s} F(x, y, z, t) = 0 \quad \text{for} \quad F(x, y, z, t) = 0, \tag{39}$$
where s is any integer. On taking $s = 2$ we have
$$\frac{D}{Dt}\left(\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}\right) = 0 \tag{40}$$
on the surface. Expanding this, we see that the time derivatives of u, v, w only occur in the form
$$\frac{\partial u}{\partial t}\frac{\partial F}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial F}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial F}{\partial z},$$
and thus have (38) expressed in the form stated.

The right-hand sides of (36) and (37) being known, the determination of P is reduced to a Neumann's problem.

7. We shall now apply the methods of the preceding section to determine the pressure immediately after the sudden generation from relative rest of any possible motion of any magnitude.

Let ϕ denote the auxiliary function defined as in § 2, which would always be the velocity-potential if the axes were at rest and the motion irrotational.

Now the initial relative motion will be irrotational, and ϕ will be its velocity-potential. Thus we have, from (36),
$$\nabla^2 P = 0, \tag{41}$$
while it follows, from what was said in the preceding section, that
$$-\frac{\partial P}{\partial n} = -\frac{\partial \phi^{\cdot}}{\partial n} + 2\omega\left(l\frac{\partial \phi}{\partial y} - m\frac{\partial \phi}{\partial x}\right). \tag{42}$$

We therefore have, initially,
$$P = \phi^{\cdot} + 2\omega P', \tag{43}$$
where P′ is determined by
$$\nabla^2 P' = 0, \tag{44}$$
in the liquid, and
$$-\frac{\partial P'}{\partial n} = l\frac{\partial \phi}{\partial y} - m\frac{\partial \phi}{\partial x}, \tag{45}$$
at the boundaries.

For the pressure components we have then

$$Q_r = Q_r' + Q_r'' + Q_r''',$$

where
$$\frac{Q_r'}{\rho} = \Sigma \iint (\dot{\phi} - \tfrac{1}{2}q^2 - \Omega) \frac{\partial \phi_r}{\partial n} dS,$$

$$\frac{Q_r''}{\rho} = 2\omega \Sigma \iint P' \frac{\partial \phi_r}{\partial n} dS$$

$$= 2\omega \Sigma \iint \phi_r \frac{\partial P'}{\partial n} dS$$

$$= -2\omega \Sigma \iint \phi_r \left(l \frac{\partial \phi}{\partial y} - m \frac{\partial \phi}{\partial x}\right) dS, \tag{47}$$

$$\frac{Q_r'''}{\rho} = -\tfrac{1}{2}\omega^2 \frac{\partial I_1}{\partial q_r}. \tag{48}$$

Here Q_r' is what the whole corresponding pressure component would be if the axes were at rest, while Q_r''' is, as in two-dimensional cases, equal to the corresponding centripetal component of mass-acceleration of solids occupying always the positions of the actual solids but of density ρ. We may write Q_r'' in the form

$$\frac{Q_r''}{2\omega\rho} = \beta_{r,1}\dot{q}_1 + \beta_{r,2}\dot{q}_2 + \ldots + \beta_{r,n}\dot{q}_n, \tag{49}$$

where
$$\beta_{r,s} = -\Sigma \iint \phi_r \left(l \frac{\partial \phi_s}{\partial y} - m \frac{\partial \phi_s}{\partial x}\right) dS$$

$$= -\Sigma \iint \phi_r \sin\theta \frac{\partial \phi_s}{\partial s} dS, \tag{50}$$

where $\partial/\partial s$ denotes differentiation along an arc of the boundary which lies in a plane perpendicular to the axis of rotation, and θ the angle between the outward-drawn normal and this axis. We notice that $\beta_{r,s} = -\beta_{s,r}$.

The coefficients $\beta_{r,s}$ are all that have values peculiar to the present problem. It is interesting to compare them with the corresponding coefficients in the gyrostatic components of the mass-acceleration of solids occupying always the positions of the actual solids, but of uniform density. These coefficients are given by

$$\beta_{r,s}' = \tfrac{1}{2}\left(\frac{\partial^2 M_0}{\partial \dot{q}_r \partial q_s} - \frac{\partial^2 M_0}{\partial \dot{q}_s \partial q_r}\right),$$

where M_0 is the moment of momentum about the axis of the relative motion of solids occupying always the positions of the actual solids, but of unit density.

8. As an example we will calculate the coefficients $\beta_{r,s}$ for the initial motion of an ellipsoid of revolution, with its axis parallel to that of rotation.

Let the co-ordinates of its centre be q_1, q_2, q_3; then we require the functions ϕ_1, ϕ_2, ϕ_3 on the surface only.

Let us first consider the case of the sphere of radius a. Take spherical polar co-ordinates r, θ, χ, with the pole at the centre of the sphere, the axis $\theta = 0$ parallel to Oz, and the plane $\chi = 0$ parallel to that of $y = 0$. Then we shall have

$$\phi_1 = \mathrm{A}a \sin\theta \cos\chi, \qquad \phi_2 = \mathrm{A}a \sin\theta \sin\chi, \qquad \phi_3 = \mathrm{A}a \cos\theta, \tag{51}$$

where $\mathrm{A} = \frac{1}{2}$, while $\sin\theta\, \partial/\partial s = \partial/a d\chi$. Therefore,

$$\beta_{1,2} = -\mathrm{A}^2 a^3 \int_{\theta=0}^{\pi} \int_{\chi=0}^{2\pi} \sin^3\theta \cos^2\chi \, d\theta \, d\chi = -\tfrac{4}{3}\pi a^3 \mathrm{A}^2, \tag{52}$$

while $$\beta_{1,3} = \beta_{2,3} = 0.$$

Let us next consider a prolate ellipsoid, of polar and equatorial radii a and b respectively. Take co-ordinates ξ, η, χ, such that if x_0, y_0, z_0 be the Cartesian co-ordinates of the centre of the ellipsoid,

$$x - x_0 = c \sinh\xi \sin\eta \cos\chi, \qquad y - y_0 = c \sinh\xi \sin\eta \sin\chi,$$
$$z - z_0 = c \cosh\xi \cos\eta,$$

where $c^2 = a^2 - b^2$.

On the surface of the ellipsoid we then have*

$$\phi_1 = \mathrm{A}b \sin\eta \cos\chi, \qquad \phi_2 = \mathrm{A}b \sin\eta \sin\chi, \qquad \phi_3 = \mathrm{C}b \cos\eta, \tag{53}$$

where $$\mathrm{A} = -\frac{\frac{1}{2}\log\{(1+e)/(1-e)\} - e/(1-e^2)}{\frac{1}{2}\log\{(1+e)/(1-e)\} - e(1-2e^2)/(1-e^2)}, \tag{54}$$

e being the eccentricity of a meridian section, so that $b^2 = a^2(1-e^2)$, and C is a similar constant, but is not required in what follows. Therefore

$$\begin{aligned} \beta_{1,2} &= -\iint \phi_1 \sin\theta \frac{\partial\phi_2}{\partial s} \, d\mathrm{S}, \\ &= -\int_{z=-a}^{a} \int_{\chi=0}^{2\pi} \phi_1 \frac{\partial\phi_2}{\partial\chi} \, dz \, d\chi \\ &= -\mathrm{A}^2 a b^2 \int_{\eta=0}^{\pi} \int_{\chi=0}^{2\pi} \sin^3\eta \cos^2\chi \, d\eta \, d\chi \\ &= -\tfrac{4}{3}\pi a b^2 \mathrm{A}^2, \end{aligned} \tag{55}$$

while $\beta_{1,3} = \beta_{2,3} = 0$.

Considering an oblate ellipsoid of polar and equatorial radii b and a respectively, similar analysis shows that

$$\beta_{1,2} = -\tfrac{4}{3}\pi a^2 b \mathrm{A}^2, \qquad \beta_{1,3} = \beta_{2,3} = 0, \tag{56}$$

where now $$\mathrm{A} = -\frac{e(1-e^2) - (1-e^2)^{\frac{1}{2}} \sin^{-1} e}{e(1+e^2) - (1-e^2)^{\frac{1}{2}} \sin^{-1} e}. \tag{57}$$

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* See, for instance, Lamb, 'Hydrodynamics,' 3rd Edit., p. 135.

Now let us compare (52), (55), and (56) with the corresponding coefficients $\beta_{r,s}'$ in the mass-acceleration of an ellipsoid of uniform density. We see that the ratio $\beta_{1,2}/\beta_{1,2}'$ is in each case A^2. Now in (51) $A = \frac{1}{2}$, in (54)

$$\begin{aligned} A &\longrightarrow \tfrac{1}{2} \quad \text{as} \quad e \longrightarrow 0, \\ A &\longrightarrow 1 \quad \text{as} \quad e \longrightarrow 1, \end{aligned}$$

while in (57)

$$\begin{aligned} A &\longrightarrow \tfrac{1}{2} \quad \text{as} \quad e \longrightarrow 0, \\ A &\longrightarrow 0 \quad \text{as} \quad e \longrightarrow 1. \end{aligned}$$

Thus for the sphere the ratio $\beta_{1,2}/\beta_{1,2}'$ is $\frac{1}{4}$; as the prolate ellipsoid tends to take a spindle shape the ratio tends to unity, which is its value for two-dimensional motion, while as the oblate ellipsoid tends to take a disc shape the ratio tends to zero. These results completely bear out the general reasoning of Taylor.

9. When the relative motion is either irrotational or small, the equations (32) become

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\omega v &= -\frac{\partial P}{\partial x}, \\ \frac{\partial v}{\partial t} + 2\omega u &= -\frac{\partial P}{\partial y}, \\ \frac{\partial w}{\partial t} \qquad\quad &= -\frac{\partial P}{\partial z}, \end{aligned}\right\} \tag{58}$$

a small motion being one in which squares and products of velocity-components may be neglected. It immediately follows that the only such motion which can remain "steady" is one in which $\partial P/\partial z = 0$—*i.e.*, a two-dimensional motion perpendicular to the axis of rotation.

As a first problem in the discussion of motion over an interval of time, let us consider the motion of a liquid inside an envelope which is fixed relatively to the rotating axes, on the assumption that the motion always remains small.

We have the equations (58) with

$$P = p/\rho + \Omega - \tfrac{1}{2}\omega^2(x^2+y^2), \tag{59}$$

and, at the boundary,

$$lu + mv + nw = 0. \tag{60}$$

Suppose the following expansions to be permissible, at least for a certain range of t:

$$u = \sum_{s=0}^{\infty} u_s \frac{(2\omega t)^s}{s\,!}, \quad v = \sum_{s=0}^{\infty} v_s \frac{(2\omega t)^s}{s\,!}, \quad w = \sum_{s=0}^{\infty} w_s \frac{(2\omega t)^s}{s\,!}, \tag{61}$$

$$P = 2\omega \sum_{s=0}^{\infty} P_s \frac{(2\omega t)^s}{s\,!}. \tag{62}$$

Here u_s, v_s, w_s and P_s are functions of x, y, z only, and u_0, v_0, w_0, which give the initial state of motion, are supposed known.

On substituting in (58) and (60) we obtain

$$\left.\begin{aligned} u_{s+1}-v_s &= -\frac{\partial P_s}{\partial x}, \\ v_{s+1}+u_s &= -\frac{\partial P_s}{\partial y}, \\ w_{s+1} \quad &= -\frac{\partial P_s}{\partial z}, \end{aligned}\right\} \tag{63}$$

with

$$lu_s+mv_s+nw_s = 0, \tag{64}$$

at the boundary, for $s = 0, 1, 2, \ldots.$ From these we derive

$$\left.\begin{aligned} u_{s+1}+u_{s-1} &= -\frac{\partial P_s}{\partial x}-\frac{\partial P_{s-1}}{\partial y}, \\ v_{s+1}+v_{s-1} &= -\frac{\partial P_s}{dy}+\frac{\partial P_{s-1}}{\partial x}, \\ w_{s+1}+w_{s-1} &= -\frac{\partial P_s}{\partial z}-\frac{\partial P_{s-2}}{\partial z}, \end{aligned}\right\} \tag{65}$$

for $s = 2, 3, 4, \ldots$, while these equations remain valid for $s = 1$ if we replace $\partial P_{-1}/\partial z$ by $-w_0$.

From (65) we have then, inside the liquid,

$$\nabla^2 P_s = -\frac{\partial^2 P_{s-2}}{\partial z^2}, \tag{66}$$

while at the boundary

$$\frac{\partial P_s}{\partial n} = -l\frac{\partial P_{s-1}}{\partial y}+m\frac{\partial P_{s-1}}{\partial x}-n\frac{\partial P_{s-2}}{\partial z}. \tag{67}$$

These are valid for $s = 2, 3, 4, \ldots.$ We have also

$$\nabla^2 P_1 = \frac{\partial w_0}{\partial z}, \tag{66'}$$

$$\frac{\partial P_1}{\partial n} = l\frac{\partial P_0}{\partial y}+m\frac{\partial P_0}{\partial x}+nw_0, \tag{67'}$$

while from (63) we have

$$\nabla^2 P_0 = \frac{\partial v_0}{\partial x}-\frac{\partial u_0}{\partial y}, \tag{66''}$$

$$\frac{\partial P_0}{\partial n} = lv_0-wu_0. \tag{67''}$$

The functions P_s are thus determined successively by the solution of a sequence of Neumann's problems. They depend only on the initial motion of the liquid and the size and shape of the boundary.

We will now express the velocity of the liquid at any time in terms of the sequence P_s.

Take
$$\gamma_s(z) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{z^{s+2n}}{(s+2n)!}; \tag{68}$$

we have
$$\gamma_s(z) + \gamma_{s+2}(z) = \frac{z^s}{s!}, \tag{69}$$

$$\frac{d}{dz}\gamma_s(z) = \gamma_{s-1}(z), \quad (s > 0), \tag{70}$$

and we notice that
$$\gamma_0(z) = \cos z, \quad \gamma_1(z) = \sin z.$$

Multiply the first equation in (63) by $\gamma_{s+1}(2\omega t)$, the second by $\gamma_{s+2}(2\omega t)$, and add for $s = 0, 1, 2, \ldots$. The coefficient of u_s is $(2\omega t)^s/s!$, by virtue of (69), for $s > 0$, while the coefficient of u_0 is $1 - \gamma_0(2\omega t)$. Also, the coefficient of v_s is zero for $s > 0$, while the coefficient of v_0 is $-\gamma_1(2\omega t)$. We therefore obtain, on using (61),

$$u = u_0\gamma_0(2\omega t) + v_0\gamma_1(2\omega t) - \frac{\partial}{\partial x}\sum_{s=0}^{\infty} P_s\gamma_{s+1}(2\omega t) - \frac{\partial}{\partial y}\sum_{s=0}^{\infty} P_s\gamma_{s+2}(2\omega t). \tag{71}$$

Similarly, we may obtain

$$v = v_0\gamma_0(2\omega t) - u_0\gamma_1(2\omega t) - \frac{\partial}{\partial y}\sum_{s=0}^{\infty} P_s\gamma_{s+1}(2\omega t) + \frac{\partial}{\partial x}\sum_{s=0}^{\infty} P_s\gamma_{s+2}(2\omega t), \tag{72}$$

$$w = w_0 - \frac{\partial}{\partial z}\sum_{s=0}^{\infty} P_s \frac{(2\omega t)^{s+1}}{(s+1)!}. \tag{73}$$

10. Methods similar to those of the preceding section may be used to discuss the motion of a body which moves irrotationally relative to the rotating axes. By this we mean that the body remains fixed relative to a set of axes which themselves always remain parallel to the fundamental rotating axes, but whose origin moves in any manner. The most general motion of a sphere may be regarded as of this type, when points on its surface are not required to be identified.

These two sets of rotating axes we shall refer to as the fundamental and the secondary axes.

Let u, v, w denote the components at time t of the velocity of the liquid relative to the fundamental axes, at a point whose co-ordinates relative to the secondary axes are x, y, z. Then, the motion being small, the equations (58) remain valid, but we have now

$$P = p/\rho + \Omega - \tfrac{1}{2}\omega^2\varpi^2, \tag{74}$$

ϖ being the distance from the axis of rotation.

Let V denote the outward normal velocity of the boundary relative to the fundamental axes. Then the boundary condition will be

$$lu + mv + nw = V.$$

Suppose now the expressions (61) and (62) to be valid, at least for a certain range of t, together with

$$\mathrm{V} = \sum_{s=0}^{\infty} \mathrm{V}_s \frac{(2\omega t)^s}{s\,!}. \tag{75}$$

Then we shall have (63), (65), (66), (66′), (66″), as before, but instead of (64) we shall have

$$lu_s + mv_s + nw_s = \mathrm{V}_s, \tag{76}$$

and therefore

$$\frac{\partial \mathrm{P}_s}{\partial n} = -\mathrm{V}_{s+1} - \mathrm{V}_{s-1} - l\frac{\partial \mathrm{P}_{s-1}}{\partial y} + m\frac{\partial \mathrm{P}_{s-1}}{\partial x} - n\frac{\partial \mathrm{P}_{s-2}}{\partial z}, \tag{77}$$

$$\frac{\partial \mathrm{P}_1}{\partial n} = -\mathrm{V}_2 - \mathrm{V}_0 - l\frac{\partial \mathrm{P}_0}{\partial y} + m\frac{\partial \mathrm{P}_0}{\partial x} + nw_0, \tag{77′}$$

$$\frac{\partial \mathrm{P}_0}{\partial n} = -\mathrm{V}_1 + lv_0 - mu_0, \tag{77″}$$

for $s > 1$, instead of (67), (67′), (67″), respectively.

The formulæ (71), (72), (73) remain valid in the present problem.

We shall now show that the functions P_s, for the most general state of motion of the type we are considering, provided only that the initial relative motion is irrotational, may be derived from those for constant translations parallel to each of the fundamental axes, suddenly generated from rest.

When the secondary axes have a unit translation parallel to Ox of the fundamental axes, suddenly generated at time $t = 0$, and then maintained, let P_s be denoted by X_s. In this case

$$\mathrm{V}_0 = l, \quad \mathrm{V}_s = 0 \quad (s > 0), \tag{78}$$

and the initial motion will be irrotational. Let X_{-1} denote the velocity-potential of the initial motion, so that

$$\nabla^2 \mathrm{X}_{-1} = 0, \qquad \frac{\partial \mathrm{X}_{-1}}{\partial n} = -l, \tag{79}$$

$$u_0 = -\frac{\partial \mathrm{X}_{-1}}{\partial n}, \qquad v_0 = -\frac{\partial \mathrm{X}_{-1}}{\partial y}, \qquad w_0 = -\frac{\partial \mathrm{X}_{-1}}{\partial z}. \tag{80}$$

Similarly, for unit translations parallel to Oy and Oz, similarly generated and maintained, let P_s be denoted by Y_s and Z_s respectively.

Suppose now that the velocity of the origin of the secondary axes has components

$$\sum_{s=0}^{\infty} a_s \frac{(2\omega t)^s}{s\,!}, \qquad \sum_{s=0}^{\infty} b_s \frac{(2\omega t)^s}{s\,!}, \qquad \sum_{s=0}^{\infty} c_s \frac{(2\omega t)^s}{s\,!}. \tag{81}$$

Then it is easily seen from an inspection of the conditions that we shall have

$$\mathrm{P}_s = \sum_{n=0}^{s+1} (a_n \mathrm{X}_{s-n} + b_n \mathrm{Y}_{s-n} + c_n \mathrm{Z}_{s-n}), \tag{82}$$

for $s = -1, 0, 1, 2, 3, \ldots$, where P_{-1} will give the velocity-potential of the initial relative motion.

It is now an easy matter to write down the equations of motion of the body, if the motion is known to be of the type we are considering.

As has already been mentioned, it is proposed in a future paper to give an account of the small motion of a sphere in a form which is numerically intelligible. The expansions used in this particular case have been proved to converge for all finite values of t.

A Theory of Colour Vision.

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At the present time there is no generally accepted theory of colour vision, and there is considerable divergence of opinion as to the tests which should be applied for colour blindness. The Young-Helmholtz theory certainly explains the facts of colour mixing, but it is doubtful whether we can deduce from these facts the existence of three primary colour sensations.

For, if we represent colours by particles on a plane, making the masses of the particles proportional to the luminosities of the colours, and represent mixtures of these colours by the rule for determining the centre of gravity of the particles, as is done on Newton's colour diagram, then, if any triangle be drawn enclosing all the possible points, its corners may be taken as representing the three primary colours, and these corners may take up an infinite number of positions. The usual procedure is to make two sides of this triangle fit the curve representing the spectrum as closely as possible, and take two corners of the triangle on the ends of the curve. Two of the primary colours are then definite colours of the spectrum, while the third is a sensation which can never be produced pure. But there is a certain amount of arbitrariness about this proceeding.

Also there is no doubt that the belief in primary colours is partly due to the experiments in mixing pigments which we are all familiar with from childhood, and rests rather on an inveterate tradition than on reason. Just as there were four elements, fire, water, earth and air, so there ought to be primary colours. I have several times demonstrated to schoolboys, who